

DEGENERATE BIFURCATION POINTS OF PERIODIC SOLUTIONS OF AUTONOMOUS HAMILTONIAN SYSTEMS

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ABSTRACT. We study connected branches of non-constant 2π -periodic solutions of the Hamilton equation

$$\dot{x}(t) = \lambda J \nabla H(x(t)),$$

where $\lambda \in (0, +\infty)$, $H \in C^2(\mathbb{R}^n \times \mathbb{R}^n, \mathbb{R})$ and $\nabla^2 H(x_0) = \begin{bmatrix} A & 0 \\ 0 & B \end{bmatrix}$ for $x_0 \in \nabla H^{-1}(0)$.

The Hessian $\nabla^2 H(x_0)$ can be singular. We formulate sufficient conditions for the existence of such branches bifurcating from given (x_0, λ_0) . As a consequence we prove theorems concerning the existence of connected branches of arbitrary periodic nonstationary trajectories of the Hamiltonian system $\dot{x}(t) = J \nabla H(x(t))$ emanating from x_0 . We describe also minimal periods of trajectories near x_0 .

1. INTRODUCTION

Consider the autonomous Hamiltonian system

$$\dot{x}(t) = J \nabla H(x(t)), \tag{1.1}$$

where $H \in C^2(\mathbb{R}^n \times \mathbb{R}^n, \mathbb{R})$ and J is the standard $2n$ -dimensional symplectic matrix. The problem of finding periodic solutions of (1.1) is equivalent to the problem of finding solutions of the family

$$\begin{cases} \dot{x}(t) = \lambda J \nabla H(x(t)) \\ x(0) = x(2\pi), \end{cases} \tag{1.2}$$

with $\lambda \in (0, +\infty)$. Let $(x_0, \lambda_0) \in \nabla H^{-1}(0) \times (0, +\infty)$, where $\nabla H^{-1}(0) \equiv (\nabla H)^{-1}(\{0\})$. Having a connected branch of nontrivial solutions of (1.2) bifurcating (in a suitable space) from (x_0, λ_0) we can find the corresponding connected branch of nonstationary periodic trajectories of (1.1) emanating from x_0 with periods tending to $2\pi\lambda_0$ at x_0 . Our aim is to study such connected branches of bifurcations and emanations when the Hessian of H at x_0 has the block-diagonal form

$$\nabla^2 H(x_0) = \begin{bmatrix} A & 0 \\ 0 & B \end{bmatrix}, \tag{1.3}$$

where A and B are real symmetric $(n \times n)$ -matrices. The critical point x_0 can be degenerate, i.e. $\nabla^2 H(x_0)$ can be singular. However, we assume that x_0 is isolated in $\nabla H^{-1}(0)$ and the Brouwer degree of ∇H around x_0 is nonzero.

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Condition (1.3) is satisfied, for example, if

$$H(x) = H(y, z) = \frac{1}{2} \langle M^{-1}y, y \rangle + V(z), \quad (1.4)$$

where $y, z \in \mathbb{R}^n$, $V \in C^2(\mathbb{R}^n, \mathbb{R})$ and M is nonsingular real symmetric $(n \times n)$ -matrix. In such a case Eq. (1.1) is equivalent to the Newton equation

$$M\ddot{z}(t) = -\nabla V(z(t)). \quad (1.5)$$

Two basic results concerning bifurcations of periodic solutions from a nondegenerate stationary point of Hamiltonian system are due to Liapunov and Berger. If $J\nabla^2 H(x_0)$ is nonsingular and has two purely imaginary eigenvalues $\pm i\beta$ of multiplicity 1 then Liapunov center theorem ensures the existence of a one-parameter family of nonstationary periodic solutions of (1.1) emanating from nondegenerate $x_0 \in \nabla H^{-1}(0)$ (see [9]). Berger [2, 3] proved the existence of a sequence of nonstationary periodic solutions convergent to the nondegenerate stationary solution of (1.5) for $M = I$, without any assumptions on multiplicity of eigenvalues of the Hessian of V (see also [4, 15]). The above results were generalized in [6, 14, 16] to the case of Hamiltonian systems with degenerate stationary points. Authors of [14, 16] used Morse theory and they obtained, similarly as Berger, sequences of periodic solutions. In [6] connected branches of nontrivial solutions of (1.2) were obtained by using the topological degree theory for $\mathbb{S}\mathbb{O}(2)$ -equivariant gradient maps (see [5]) and the results from [11]. Global bifurcation theorems of this type can be also found in [7, 13]. In the present paper we apply results from [6] to prove the existence of connected branches of bifurcations and emanations in degenerate case under assumptions written in terms of the topological degree of ∇H around x_0 and eigenvalues of $\nabla^2 H(x_0)$. In particular, we generalize results from [12], proved for H satisfying (1.3) with $A = I$.

2. PRELIMINARIES

In this section we set up notation and summarize without proofs the relevant material on bifurcation theory for Hamiltonian systems.

The number of elements of any finite set X will be denoted as $\sharp X$. Write $\mathbb{M}(n, \mathbb{R})$ for the set of real $(n \times n)$ -matrices. Let $I_n \in \mathbb{M}(n, \mathbb{R})$ stand for the identity matrix. Define $J_n \in \mathbb{M}(2n, \mathbb{R})$ as

$$J_n = \begin{bmatrix} 0 & -I_n \\ I_n & 0 \end{bmatrix}.$$

Usually we abbreviate I_n and J_n to I and J . Write $\mathbb{S}(n, \mathbb{R})$, $\mathbb{O}(n, \mathbb{R})$ and $\mathbb{GL}(n, \mathbb{R})$ for the subsets of $\mathbb{M}(n, \mathbb{R})$ consisting of symmetric, orthogonal and nonsingular matrices, respectively. If $\alpha_1, \dots, \alpha_n \in \mathbb{R}$ then $\text{diag}(\alpha_1, \dots, \alpha_n)$ denotes the diagonal matrix with $\alpha_1, \dots, \alpha_n$ on the main diagonal.

Let $\sigma(A)$ be the spectrum of $A \in \mathbb{M}(n, \mathbb{R})$. Obviously, if $A \in \mathbb{S}(n, \mathbb{R})$ then $\sigma(A) \subset \mathbb{R}$. Symbols $\sigma_+(A)$ and $\sigma_-(A)$ denote the sets of strictly positive and strictly negative real eigenvalues of A , respectively. Let $\mu(\alpha)$ be the multiplicity of the eigenvalue $\alpha \in \sigma(A)$ and write $V_A(\alpha)$ for the eigenspace corresponding to α . For symmetric A we define the

Morse index $m^-(A)$ of A by the formula

$$m^-(A) = \sum_{\alpha \in \sigma_-(A)} \mu(\alpha).$$

If $B \in \mathbb{S}(n, \mathbb{R})$ is nonnegative definite then there exists a nonnegative definite $C \in \mathbb{S}(n, \mathbb{R})$ such that $C^2 = B$. We denote C by \sqrt{B} . For any $A \in \mathbb{S}(n, \mathbb{R})$ write $|A| = \sqrt{A^2}$. If A is nonnegative or nonpositive definite then we write

$$\text{sgn}(A) = \begin{cases} 1; & \sigma_-(A) = \emptyset, A \neq 0, \\ 0; & A = 0, \\ -1; & \sigma_+(A) = \emptyset, A \neq 0. \end{cases}$$

In such a case $A = \text{sgn}(A) |A|$.

For any $K, L \in \mathbb{S}(2n, \mathbb{R})$ and $j \in \mathbb{N}$ set

$$Q_j(K) = \begin{bmatrix} -K & jJ^t \\ jJ & -K \end{bmatrix},$$

$$\Lambda_j(L) = \{\lambda \in (0, +\infty) \mid \det Q_j(\lambda L) = 0\},$$

$$\Lambda(L) = \bigcup_{j \in \mathbb{N}} \Lambda_j(L).$$

Obviously, $Q_j(K) \in \mathbb{S}(4n, \mathbb{R})$ and since $\Lambda_j(L)$ is a subset of roots of a polynomial, it is finite. Moreover, observe that the following lemma holds true.

Lemma 2.1. *For every $j \in \mathbb{N}$ we have*

$$\Lambda_j(L) = j\Lambda_1(L) \equiv \{j\alpha \mid \alpha \in \Lambda_1(L)\}.$$

In particular, for every $a, b \in \mathbb{R}$, $a < b$, the set $\Lambda(L) \cap [a, b]$ is finite.

In what follows we assume that $H \in C^2(\mathbb{R}^n \times \mathbb{R}^n, \mathbb{R})$. We call $\mathcal{T}(H) = \nabla H^{-1}(0) \times (0, +\infty)$ the set of *trivial solutions* of (1.2). The set $\mathcal{NT}(H)$ of *nontrivial solutions* of (1.2) consists of those solutions (x, λ) of (1.2) that do not belong to $\mathcal{T}(H)$. We will consider $\mathcal{T}(H)$ and $\mathcal{NT}(H)$ as subsets of $H_{2\pi}^1 \times (0, +\infty)$, see [10] for the definition of the Hilbert space $H_{2\pi}^1 \equiv H^1([0, 2\pi], \mathbb{R}^{2n})$. (Recall that $x(0) = x(2\pi)$ for every $x \in H_{2\pi}^1$.) The trivial solution $(x_0, \lambda_0) \in \mathcal{T}(H)$ is said to be a *bifurcation point of nontrivial solutions* of (1.2) if it is a cluster point of $\mathcal{NT}(H)$. We say that a set $C \subset \mathcal{NT}(H)$ *bifurcates* from (x_0, λ_0) if $(x_0, \lambda_0) \in \text{cl}(C)$. Denote by $C(x_0, \lambda_0)$ the connected component of $\text{cl}(\mathcal{NT}(H))$ containing the bifurcation point (x_0, λ_0) . If $C(x_0, \lambda_0) \neq \{(x_0, \lambda_0)\}$ then (x_0, λ_0) is called a *branching point of nontrivial solutions* of (1.2). It was proved in [6] that any point $(x, 0)$, $x \in H_{2\pi}^1$, cannot be a cluster point of $\mathcal{NT}(H)$ in $H_{2\pi}^1 \times \mathbb{R}$.

The following theorem (see [6]) gives a necessary condition for (x_0, λ_0) to be a bifurcation point.

Theorem 2.2. *Let x_0 be isolated in $\nabla H^{-1}(0)$. If $(x_0, \lambda_0) \in \mathcal{T}(H)$ is a bifurcation point of nontrivial solutions of (1.2) then $\lambda_0 \in \Lambda(\nabla^2 H(x_0))$.*

Fix isolated $x_0 \in \nabla H^{-1}(0)$, $\lambda_0 \in \Lambda(\nabla^2 H(x_0))$ and choose $\varepsilon > 0$ such that $\lambda_0 - \varepsilon > 0$ and $\Lambda(\nabla^2 H(x_0)) \cap [\lambda_0 - \varepsilon, \lambda_0 + \varepsilon] = \{\lambda_0\}$ (see Lemma 2.1). Define the *bifurcation index* $\eta(x_0, \lambda_0) = \{\eta_j(x_0, \lambda_0)\}_{j \in \mathbb{N}}$ (originally defined by using topological degree for $\mathbb{S}\mathbb{O}(2)$ -equivariant gradient mappings, see [5, 6]) as follows:

$$\eta_j(x_0, \lambda_0) = i(\nabla H, x_0) \cdot \frac{m^-(Q_j((\lambda_0 + \varepsilon)\nabla^2 H(x_0))) - m^-(Q_j((\lambda_0 - \varepsilon)\nabla^2 H(x_0)))}{2}, \quad (2.1)$$

where $i(\nabla H, x_0)$ is the topological index of x_0 with respect to ∇H , i.e. it is the Brouwer degree of ∇H on the neighbourhood Ω of x_0 such that $\text{cl}(\Omega) \cap \nabla H^{-1}(0) = \{x_0\}$.

Let $\Theta \equiv (0, 0, \dots)$. The following theorem gives a sufficient condition for a trivial solution to be a branching point.

Theorem 2.3. *Fix isolated $x_0 \in \nabla H^{-1}(0)$ and $\lambda_0 \in \Lambda(\nabla^2 H(x_0))$. If $\eta(x_0, \lambda_0) \neq \Theta$ then (x_0, λ_0) is a branching point of nontrivial solutions of (1.2). Moreover, if $U = \Omega \times (a, b) \subset H_{2\pi}^1 \times (0, +\infty)$ is an open bounded neighbourhood of (x_0, λ_0) such that $\nabla H^{-1}(0) \cap \text{cl}(\Omega) = \{x_0\}$ and $\Lambda(\nabla^2 H(x_0)) \cap [a, b] = \{\lambda_0\}$ then $C(x_0, \lambda_0) \cap \partial U \neq \emptyset$.*

The proof of Theorem 2.3 can be obtained by a slight modification of the proof of the following Rabinowitz-type global bifurcation theorem for Hamiltonian systems, see [6].

Theorem 2.4. *Let $\nabla H^{-1}(0)$ be finite. Fix $x_0 \in \nabla H^{-1}(0)$, $\lambda_0 \in \Lambda(\nabla^2 H(x_0))$. If $\eta(x_0, \lambda_0) \neq \Theta$ then (x_0, λ_0) is a branching point of nontrivial solutions of (1.2), and either*

- (1) $C(x_0, \lambda_0)$ is unbounded in $H_{2\pi}^1 \times (0, +\infty)$ or
- (2) $C(x_0, \lambda_0)$ is bounded and, in addition, $C(x_0, \lambda_0) \cap \mathcal{T}(H) = \{y_1, \dots, y_m\}$ for some $m \in \mathbb{N}$, $y_1, \dots, y_m \in \mathcal{T}(H)$, and

$$\sum_{i=1}^m \eta(y_i) = \Theta.$$

We will regard the set $\mathcal{S}(H) = \nabla H^{-1}(0)$ of *stationary solutions* of (1.1) and the set $\mathcal{NS}(H)$ of *nonstationary periodic solutions* as subsets of the Banach space $B_0 \equiv B_0(\mathbb{R}, \mathbb{R}^{2n})$ of bounded functions with the supremum norm denoted as $\|\cdot\|_0$. If the stationary point $x_0 \in \nabla H^{-1}(0)$ is a cluster point of $\mathcal{NS}(H)$ then it is said to be an *emanation point of nonstationary periodic solutions* of (1.1).

Denote by $\mathcal{M}_{\mathcal{H}} \equiv \mathcal{M}_{\mathcal{H}}(\mathbb{R}^{2n})$ the complete metric space of nonempty compact subsets of \mathbb{R}^{2n} with Hausdorff metric $d_{\mathcal{H}}$ defined by the formula

$$d_{\mathcal{H}}(A, B) = \max \left\{ \sup_{a \in A} \text{dist}(a, B), \sup_{b \in B} \text{dist}(b, A) \right\}$$

for $A, B \in \mathcal{M}_{\mathcal{H}}$. For every $x_0 \in \nabla H^{-1}(0)$, $x \in \mathcal{NS}(H)$ we have $\|x - x_0\|_0 = d_{\mathcal{H}}(\text{tr}(x), x_0)$, where $\text{tr}(x)$ is a trajectory of x . Thus x_0 is an emanation point of nonstationary periodic solutions iff it is an emanation point (in $\mathcal{M}_{\mathcal{H}}$) of nonstationary periodic trajectories. We say that a set $C \subset \mathcal{M}_{\mathcal{H}}(\mathbb{R}^{2n})$ of nonstationary periodic trajectories of (1.1) *emanates* from x_0 if $x_0 \in \text{cl}(C)$ (in $\mathcal{M}_{\mathcal{H}}(\mathbb{R}^{2n})$). Note that if C is connected with respect to Hausdorff metric then the union of trajectories from C is a connected subset of \mathbb{R}^{2n} .

Remarks 2.5. Let x_0 be isolated in $\nabla H^{-1}(0)$.

(1) Bifurcations of nontrivial solutions of (1.2) can be translated into emanations of nonstationary periodic trajectories of (1.1). Namely, if $(x, \lambda) \in H_{2\pi}^1 \times (0, +\infty)$ is a solution of (1.2) then the function $\tilde{x}_\lambda \in B_0$, $\tilde{x}_\lambda(t) = x(\frac{t}{\lambda})$, is a solution of (1.1) with (not necessarily minimal) period $2\pi\lambda$. (Notice that in this case x can be regarded as 2π -periodic function of class C^1 defined on \mathbb{R} .) Since the mapping $P : \mathcal{T}(H) \cup \mathcal{NT}(H) \rightarrow \mathcal{M}_H$, $P(x, \lambda) = \text{tr}(\tilde{x}_\lambda) = \text{tr}(x) = x([0, 2\pi])$, is continuous, for given connected branch $C \subset C(x_0, \lambda_0) \cap \mathcal{NT}(H)$ bifurcating from a trivial solution (x_0, λ_0) of (1.2) we obtain a connected branch $P(C)$ of nonstationary periodic trajectories of (1.1) emanating from x_0 .

(2) It can be proved that there exists a neighbourhood $U \subset H_{2\pi}^1 \times (0, +\infty)$ of (x_0, λ_0) such that (not necessarily minimal) periods $2\pi\lambda$ of trajectories $\text{tr}(\tilde{x}_\lambda) \in P(C(x_0, \lambda_0) \cap U)$ are arbitrarily close to $2\pi\lambda_0$ for trajectories sufficiently close to x_0 , i.e. for every $\varepsilon > 0$ there exists $\delta > 0$ such that if $(x, \lambda) \in C(x_0, \lambda_0) \cap U$, $\|\tilde{x}_\lambda - x_0\|_0 < \delta$ (or, equivalently, $d_H(\text{tr}(\tilde{x}_\lambda), x_0) < \delta$) then $|2\pi\lambda - 2\pi\lambda_0| < \varepsilon$, see [12].

(3) Moreover, if $\eta_j(x_0, \lambda_0) \neq 0$ then $C(x_0, \lambda_0) \cap \mathcal{NT}(H)$ contains a connected subset C_j bifurcating from (x_0, λ_0) such that for every $(x, \lambda) \in C_j$ the number $\frac{2\pi}{j}$ is a period (not necessarily minimal) of x , which follows from the theory of topological degree for $\mathbb{S}\mathbb{O}(2)$ -equivariant gradient mappings. Consequently, (not necessarily minimal) periods $\frac{2\pi\lambda}{j}$ of trajectories $\text{tr}(\tilde{x}_\lambda) \in P(C_j \cap U)$ tend to $\frac{2\pi\lambda_0}{j}$ at x_0 .

(4) If $\frac{\lambda_0}{\lambda} \notin \mathbb{N}$ for all $\lambda \in \Lambda(\nabla^2 H(x_0)) \setminus \{\lambda_0\}$ then U can be chosen in such a way that for every $(x, \lambda) \in C(x_0, \lambda_0) \cap U$, $x \neq x_0$, the minimal period of \tilde{x}_λ is equal to $2\pi\lambda$, see [12].

3. ALGEBRAIC RESULTS

The aim of this section is to prove some algebraic lemmas which will be used for formulation of bifurcation theorems.

Fix some $C, D \in \mathbb{S}(n, \mathbb{R})$ and $K \in \mathbb{S}(2n, \mathbb{R})$ of the form

$$K = \begin{bmatrix} C & 0 \\ 0 & D \end{bmatrix}.$$

For every $j \in \mathbb{N}$ define $G_j(K) \in \mathbb{S}(2n, \mathbb{R})$ and $X \in \mathbb{O}(4n, \mathbb{R})$ by the formulas

$$G_j(K) = \begin{bmatrix} -C & jI \\ jI & -D \end{bmatrix} = -K + \begin{bmatrix} 0 & jI \\ jI & 0 \end{bmatrix}, \quad X = \begin{bmatrix} I & 0 & 0 & 0 \\ 0 & 0 & 0 & -I \\ 0 & 0 & I & 0 \\ 0 & I & 0 & 0 \end{bmatrix},$$

where $I \equiv I_n$.

Lemma 3.1. *For any $j \in \mathbb{N}$ we have*

$$X^t Q_j(K) X = \begin{bmatrix} G_j(K) & 0 \\ 0 & G_j(K) \end{bmatrix},$$

$\dim \ker G_j(K) = \dim \ker [CD - j^2 I]$ and $\det G_j(K) = \det [CD - j^2 I]$.

Proof. The first equality of the lemma can be checked by direct calculation. Note that the matrix

$$Y = \begin{bmatrix} -j^{-1}I & -D \\ 0 & -jI \end{bmatrix}$$

is nonsingular and, moreover, $\det Y = 1$. Thus

$$\dim \ker G_j(K) = \dim \ker (G_j(K)Y) = \dim \ker \begin{bmatrix} j^{-1}C & CD - j^2I \\ -I & 0 \end{bmatrix} = \dim \ker [CD - j^2I]$$

and, similarly,

$$\det G_j(K) = \det (G_j(K)Y) = \det \begin{bmatrix} j^{-1}C & CD - j^2I \\ -I & 0 \end{bmatrix} = \det [CD - j^2I].$$

□

In what follows we assume that $A, B \in \mathbb{S}(n, \mathbb{R})$, $\sigma(A) = \{\alpha_1, \dots, \alpha_n\}$, $\sigma(B) = \{\beta_1, \dots, \beta_n\}$, and

$$L = \begin{bmatrix} A & 0 \\ 0 & B \end{bmatrix}.$$

Remark 3.2. If $AB = BA$ then there exists $E \in \mathbb{O}(n, \mathbb{R})$ which diagonalizes both A and B . Thus we may assume without loss of generality that

$$E^t A E = J(A) = \text{diag}(\alpha_1, \dots, \alpha_n), \quad E^t B E = J(B) = \text{diag}(\beta_1, \dots, \beta_n),$$

$$E^t A B E = J(AB) = J(A)J(B) = \text{diag}(\alpha_1\beta_1, \dots, \alpha_n\beta_n).$$

We use this order of α_k, β_k in the whole paper whenever A and B commute.

Lemma 3.3. *For any $j \in \mathbb{N}$ we have*

$$\Lambda_j(L) = \{\lambda \in (0, +\infty) \mid \det G_j(\lambda L) = 0\} = \left\{ \frac{j}{\sqrt{\nu}} \mid \nu \in \sigma_+(AB) \right\}.$$

Moreover, for any fixed $\nu_0 \in \sigma_+(AB)$, $j_0 \in \mathbb{N}$, the multiplicity of the root $\lambda_0 = \frac{j_0}{\sqrt{\nu_0}}$ of the polynomial $\det G_{j_0}(\lambda L)$ is equal to $\dim \ker G_{j_0}(\lambda_0 L) = \mu(\nu_0) \leq n$.

Proof. From Lemma 3.1 we obtain $\det Q_j(\lambda L) = (\det G_j(\lambda L))^2$ and

$$\det G_j(\lambda L) = \det[\lambda^2 AB - j^2 I] = \lambda^{2n} \det \left[AB - \frac{j^2}{\lambda^2} I \right]. \quad (3.1)$$

Thus $\det G_j(\lambda L) = 0$ for $\lambda \in (0, +\infty)$ if and only if $\frac{j^2}{\lambda^2} = \nu$ for some $\nu \in \sigma_+(AB)$. The equality (3.1) implies also that the multiplicity of the root $\lambda_0 = \frac{j_0}{\sqrt{\nu_0}}$ is equal to $\mu(\nu_0)$. On the other hand, by Lemma 3.1 we have $\dim \ker G_{j_0}(\lambda_0 L) = \dim \ker [AB - \frac{j_0^2}{\lambda_0^2} I] = \mu(\nu_0) \leq n$. □

Lemma 3.4. *If $AB = BA$ then*

$$\Lambda_j(L) = \left\{ \frac{j}{\sqrt{\alpha_k \beta_k}} \mid \alpha_k \beta_k > 0, k \in \{1, \dots, n\} \right\}$$

for all $j \in \mathbb{N}$.

Proof. By Remark 3.2, $\sigma(AB) = \{\alpha_k \beta_k \mid k \in \{1, \dots, n\}\}$. Applying Lemma 3.3 we complete the proof. □

Lemma 3.5. *If $\alpha_{k_0}\beta_{k_0} > 0$ and $V_A(\alpha_{k_0}) \cap V_B(\beta_{k_0}) \neq \{0\}$ for some $k_0 \in \{1, \dots, n\}$ then $\frac{j}{\sqrt{\alpha_{k_0}\beta_{k_0}}} \in \Lambda_j(L)$ for every $j \in \mathbb{N}$.*

Proof. It is easy to see that $\nu_0 = \alpha_{k_0}\beta_{k_0}$ is a strictly positive eigenvalue of AB , therefore our claim is a consequence of Lemma 3.3. \square

From now on for given $\lambda_0 \in \Lambda(L)$ we choose $\varepsilon > 0$ such that $\lambda_0 - \varepsilon > 0$ and $\Lambda(L) \cap [\lambda_0 - \varepsilon, \lambda_0 + \varepsilon] = \{\lambda_0\}$ (see Lemma 2.1).

Let

$$\begin{aligned} Y_j(\lambda_0) &= \left\{ k \in \{1, \dots, n\} \mid \alpha_k \beta_k > 0, \lambda_0 = \frac{j}{\sqrt{\alpha_k \beta_k}} \right\}, \\ Y_j^+(\lambda_0) &= \{k \in Y_j(\lambda_0) \mid \alpha_k > 0, \beta_k > 0\}, \\ Y_j^-(\lambda_0) &= \{k \in Y_j(\lambda_0) \mid \alpha_k < 0, \beta_k < 0\}. \end{aligned} \quad (3.2)$$

For $k \in \{1, \dots, n\}$ and $j \in \mathbb{N}$ define functions $\gamma_{kj}^\pm : (0, +\infty) \rightarrow \mathbb{R}$ by the formula

$$\gamma_{kj}^\pm(\lambda) = \frac{-\lambda(\alpha_k + \beta_k) \pm \sqrt{\lambda^2(\alpha_k - \beta_k)^2 + 4j^2}}{2}.$$

Let us gather some basic properties of γ_{kj}^\pm .

Lemma 3.6. *For every $k \in \{1, \dots, n\}$, $j \in \mathbb{N}$ we have*

$$\begin{aligned} \gamma_{kj}^+(\lambda) = 0 \wedge \lambda \in (0, +\infty) &\Leftrightarrow \alpha_k > 0 \wedge \beta_k > 0 \wedge \lambda = \frac{j}{\sqrt{\alpha_k \beta_k}}, \\ \gamma_{kj}^-(\lambda) = 0 \wedge \lambda \in (0, +\infty) &\Leftrightarrow \alpha_k < 0 \wedge \beta_k < 0 \wedge \lambda = \frac{j}{\sqrt{\alpha_k \beta_k}}. \end{aligned}$$

Moreover, for every fixed $\lambda_0 = \frac{j}{\sqrt{\alpha_k \beta_k}}$ we have

$$\begin{aligned} \alpha_k > 0 \wedge \beta_k > 0 &\Rightarrow \begin{cases} \gamma_{kj}^+(\lambda_0 + \varepsilon) < 0 \wedge \gamma_{kj}^+(\lambda_0 - \varepsilon) > 0 \\ \gamma_{kj}^-(\lambda_0 + \varepsilon) < 0 \wedge \gamma_{kj}^-(\lambda_0 - \varepsilon) < 0, \end{cases} \\ \alpha_k < 0 \wedge \beta_k < 0 &\Rightarrow \begin{cases} \gamma_{kj}^+(\lambda_0 + \varepsilon) > 0 \wedge \gamma_{kj}^+(\lambda_0 - \varepsilon) > 0 \\ \gamma_{kj}^-(\lambda_0 + \varepsilon) > 0 \wedge \gamma_{kj}^-(\lambda_0 - \varepsilon) < 0. \end{cases} \end{aligned}$$

Proof. First observe that

$$\gamma_{kj}^+(\lambda) \gamma_{kj}^-(\lambda) = \lambda^2 \alpha_k \beta_k - j^2, \quad (3.3)$$

therefore $\gamma_{kj}^+(\lambda) \gamma_{kj}^-(\lambda) = 0$ iff $\alpha_k \beta_k > 0$ and $\lambda = \lambda_0 = \frac{j}{\sqrt{\alpha_k \beta_k}}$. If $\alpha_k > 0$ and $\beta_k > 0$ then $\gamma_{kj}^-(\lambda_0) < 0$, hence $\gamma_{kj}^+(\lambda_0) = 0$. For $\alpha_k < 0$, $\beta_k < 0$ we have $\gamma_{kj}^+(\lambda_0) > 0$ and thus $\gamma_{kj}^-(\lambda_0) = 0$. Moreover, the derivation of (3.3) gives

$$2\lambda_0 \alpha_k \beta_k = (\gamma_{kj}^+(\lambda_0))' \gamma_{kj}^-(\lambda_0) + \gamma_{kj}^+(\lambda_0) (\gamma_{kj}^-)'(\lambda_0),$$

and thus

$$\begin{aligned} \alpha_k > 0 \wedge \beta_k > 0 &\Rightarrow (\gamma_{kj}^+)'(\lambda_0) < 0, \\ \alpha_k < 0 \wedge \beta_k < 0 &\Rightarrow (\gamma_{kj}^-)'(\lambda_0) > 0. \end{aligned}$$

\square

Lemma 3.7. *If $AB = BA$ then for every $j \in \mathbb{N}$, $\lambda \in (0, +\infty)$ and fixed $\lambda_0 \in \Lambda(L)$ we have*

$$\sigma(Q_j(\lambda L)) = \sigma(G_j(\lambda L)) = \bigcup_{k=1}^n \{\gamma_{kj}^+(\lambda), \gamma_{kj}^-(\lambda)\}$$

and

$$m^-(Q_j((\lambda_0 + \varepsilon)L)) - m^-(Q_j((\lambda_0 - \varepsilon)L)) = 2(\#Y_j^+(\lambda_0) - \#Y_j^-(\lambda_0)).$$

Proof. Set $C = \lambda A$, $D = \lambda B$. In view of Lemma 3.1, $\sigma(Q_j(\lambda L)) = \sigma(G_j(\lambda L))$ and $m^-(Q_j((\lambda_0 + \varepsilon)L)) - m^-(Q_j((\lambda_0 - \varepsilon)L)) = 2(m^-(G_j((\lambda_0 + \varepsilon)L)) - m^-(G_j((\lambda_0 - \varepsilon)L)))$. Moreover, by Remark 3.2 there is $E \in \mathbb{O}(n, \mathbb{R})$ such that

$$E^t A E = \text{diag}(\alpha_1, \dots, \alpha_n), \quad E^t B E = \text{diag}(\beta_1, \dots, \beta_n).$$

From Lemma 3.1 we obtain

$$\begin{aligned} \det[G_j(\lambda L) - \omega I] &= \det G_j(\lambda L + \omega I) = \det[(\lambda A + \omega I)(\lambda B + \omega I) - j^2 I] = \\ &= \det[(\lambda E^t A E + \omega I)(\lambda E^t B E + \omega I) - j^2 I] = \prod_{k=1}^n ((\lambda \alpha_k + \omega)(\lambda \beta_k + \omega) - j^2). \end{aligned}$$

But

$$(\lambda \alpha_k + \omega)(\lambda \beta_k + \omega) - j^2 = (\omega - \gamma_{kj}^+(\lambda))(\omega - \gamma_{kj}^-(\lambda))$$

and thus

$$\sigma(G_j(\lambda L)) = \bigcup_{k=1}^n \{\gamma_{kj}^+(\lambda), \gamma_{kj}^-(\lambda)\}.$$

To compute the change of the Morse index of $G_j(\lambda L)$ observe that according to Lemma 3.6 the eigenvalues $\gamma_{kj}^\pm(\lambda)$ may change their signs at λ_0 only if $k \in Y_j(\lambda_0)$, i.e. $\alpha_k \beta_k > 0$ and $\lambda_0 = \frac{j}{\sqrt{\alpha_k \beta_k}}$. Using this property, the definition of the Morse index, the definitions of $Y_j(\lambda_0)$, $Y_j^\pm(\lambda_0)$, and the results of Lemma 3.6 we obtain

$$\begin{aligned} m^-(G_j((\lambda_0 + \varepsilon)L)) - m^-(G_j((\lambda_0 - \varepsilon)L)) &= \\ &= \#\{k \in \{1, \dots, n\} \mid \gamma_{kj}^+(\lambda_0 + \varepsilon) < 0\} + \#\{k \in \{1, \dots, n\} \mid \gamma_{kj}^-(\lambda_0 + \varepsilon) < 0\} - \\ &- \#\{k \in \{1, \dots, n\} \mid \gamma_{kj}^+(\lambda_0 - \varepsilon) < 0\} - \#\{k \in \{1, \dots, n\} \mid \gamma_{kj}^-(\lambda_0 - \varepsilon) < 0\} = \\ &= \#\{k \in Y_j(\lambda_0) \mid \gamma_{kj}^+(\lambda_0 + \varepsilon) < 0\} + \#\{k \in Y_j(\lambda_0) \mid \gamma_{kj}^-(\lambda_0 + \varepsilon) < 0\} - \\ &- \#\{k \in Y_j(\lambda_0) \mid \gamma_{kj}^+(\lambda_0 - \varepsilon) < 0\} - \#\{k \in Y_j(\lambda_0) \mid \gamma_{kj}^-(\lambda_0 - \varepsilon) < 0\} = \\ &= \#Y_j^+(\lambda_0) + \#Y_j^-(\lambda_0) - 0 - \#(Y_j^+(\lambda_0) \cup Y_j^-(\lambda_0)) = \#Y_j^+(\lambda_0) - \#Y_j^-(\lambda_0), \end{aligned}$$

since

$$\{k \in Y_j(\lambda_0) \mid \gamma_{kj}^+(\lambda_0 - \varepsilon) < 0\} = \emptyset$$

and

$$\{k \in Y_j(\lambda_0) \mid \gamma_{kj}^-(\lambda_0 - \varepsilon) < 0\} = Y_j^+(\lambda_0) \cup Y_j^-(\lambda_0).$$

□

Lemma 3.8. *If $\alpha_{k_0}\beta_{k_0} > 0$ and $V_A(\alpha_{k_0}) \cap V_B(\beta_{k_0}) \neq \{0\}$ for some $k_0 \in \{1, \dots, n\}$ then $\gamma_{k_0j}^\pm(\lambda) \in \sigma(Q_j(\lambda L)) = \sigma(G_j(\lambda L))$ for every $j \in \mathbb{N}$, $\lambda \in (0, +\infty)$ and we have*

$$\mu(\gamma_{k_0j}^\pm(\lambda)) \geq \dim V_A(\alpha_{k_0}) \cap V_B(\beta_{k_0})$$

(where $\mu(\gamma_{k_0j}^\pm(\lambda))$ is the multiplicity of $\gamma_{k_0j}^\pm(\lambda)$ as an eigenvalue of $G_j(\lambda L)$). If, in addition,

$$\dim V_A(\alpha_{k_0}) \cap V_B(\beta_{k_0}) > \frac{1}{2}\mu(\nu_0)$$

for $\nu_0 = \alpha_{k_0}\beta_{k_0} \in \sigma_+(AB)$ then for every fixed $j_0 \in \mathbb{N}$ and $\lambda_0 = \frac{j_0}{\sqrt{\nu_0}}$ we have

$$m^-(Q_{j_0}((\lambda_0 + \varepsilon)L)) - m^-(Q_{j_0}((\lambda_0 - \varepsilon)L)) \neq 0.$$

Proof. For abbreviation of notation put $q = \dim V_A(\alpha_{k_0}) \cap V_B(\beta_{k_0})$. Let $E \in \mathbb{O}(n, \mathbb{R})$ be such that

$$E^t A E = \begin{bmatrix} \alpha_{k_0} I_q & 0 \\ 0 & \tilde{A} \end{bmatrix}, \quad E^t B E = \begin{bmatrix} \beta_{k_0} I_q & 0 \\ 0 & \tilde{B} \end{bmatrix}, \quad \tilde{A}, \tilde{B} \in \mathbb{S}(n - q, \mathbb{R}).$$

Using Lemma 3.1, similarly as in the proof of Lemma 3.7, we obtain

$$\begin{aligned} \det[G_{j_0}(\lambda L) - \omega I] &= \det G_{j_0}(\lambda L + \omega I) = \\ &= \det[(\lambda A + \omega I)(\lambda B + \omega I) - j^2 I] = \det[(\lambda E^t A E + \omega I)(\lambda E^t B E + \omega I) - j^2 I] = \\ &= ((\lambda \alpha_{k_0} + \omega)(\lambda \beta_{k_0} + \omega) - j^2)^q \cdot \det[(\lambda \tilde{A} + \omega I)(\lambda \tilde{B} + \omega I) - j^2 I] = \\ &= (\omega - \gamma_{k_0j}^+(\lambda))^q (\omega - \gamma_{k_0j}^-(\lambda))^q \cdot \det[(\lambda \tilde{A} + \omega I)(\lambda \tilde{B} + \omega I) - j^2 I] \end{aligned}$$

and thus $\gamma_{k_0j}^\pm(\lambda) \in \sigma(G_{j_0}(\lambda L))$ with $\mu(\gamma_{k_0j}^\pm(\lambda)) \geq q$.

Now, fix $j_0 \in \mathbb{N}$ and $\lambda_0 = \frac{j_0}{\sqrt{\nu_0}}$. According to Lemma 3.6, if $\alpha_{k_0} > 0$, $\beta_{k_0} > 0$ ($\alpha_{k_0} < 0$, $\beta_{k_0} < 0$) then the eigenvalue $\gamma_{k_0j_0}^+(\lambda)$ (resp. $\gamma_{k_0j_0}^-(\lambda)$) changes its sign from positive to negative (resp. from negative to positive) when λ changes from $\lambda_0 - \varepsilon$ to $\lambda_0 + \varepsilon$ (and $\gamma_{k_0j_0}^-(\lambda)$ (resp. $\gamma_{k_0j_0}^+(\lambda)$) is nonzero). Thus we have at least q eigenvalues of $G_{j_0}(\lambda L)$ changing their signs in the same way at λ_0 . The number of other eigenvalues changing their signs at λ_0 is less than q because $q > \frac{1}{2}\mu(\nu_0)$ and $\mu(\nu_0)$ is the maximal number of eigenvalues of $G_{j_0}(\lambda L)$ which may change their signs at λ_0 , since $\dim \ker G_{j_0}(\lambda_0 L) = \mu(\nu_0)$, according to Lemma 3.3. \square

Lemma 3.9. *If $\nu_0 \in \sigma_+(AB)$ is of odd multiplicity and $\lambda_0 = \frac{j_0}{\sqrt{\nu_0}}$ then*

$$m^-(Q_{j_0}((\lambda_0 + \varepsilon)L)) - m^-(Q_{j_0}((\lambda_0 - \varepsilon)L)) \neq 0.$$

Proof. In view of Lemma 3.1 it suffices to show that the product of eigenvalues of $G_{j_0}(\lambda L)$ changes its sign at λ_0 . However, according to Lemma 3.3, this product is equal to $\det G_{j_0}(\lambda L) = (\lambda - \lambda_0)^{2m+1} \psi(\lambda)$, where $2m+1$ is the multiplicity of ν_0 (for some $m \in \mathbb{N} \cup \{0\}$) and $\psi(\lambda_0) \neq 0$, which completes the proof. \square

Lemma 3.10. *If A or B is strictly positive or strictly negative definite, $\nu_0 \in \sigma_+(AB)$ and $\lambda_0 = \frac{j_0}{\sqrt{\nu_0}}$ then for every $j \in \mathbb{N}$ we have*

$$m^-(Q_j((\lambda_0 + \varepsilon)L)) - m^-(Q_j((\lambda_0 - \varepsilon)L)) =$$

$$= \begin{cases} 2 \cdot s \cdot \mu(\nu) & \text{if } \lambda_0 = \frac{j}{\sqrt{\nu}} \text{ for some } \nu \in \sigma_+(AB), \\ 0 & \text{if } \lambda_0 \neq \frac{j}{\sqrt{\nu}} \text{ for all } \nu \in \sigma_+(AB), \end{cases}$$

where $s = 1$ if A or B is strictly positive definite and $s = -1$ if A or B is strictly negative definite.

Proof. Let A be strictly positive or strictly negative definite and

$$M = \begin{bmatrix} (\sqrt{|A|})^{-1} & 0 \\ 0 & \sqrt{|A|} \end{bmatrix}.$$

Note that $M \in \mathbb{S}(2n, \mathbb{R}) \cap \mathbb{GL}(2n, \mathbb{R})$. In view of Sylvester's law of inertia, any nonsingular transformation does not change the Morse index of the symmetric matrix, hence

$$\begin{aligned} m^-(G_j(\lambda L)) &= m^-(M^t G_j(\lambda L) M) = m^-(M G_j(\lambda L) M) = \\ &= m^-\left(\begin{bmatrix} -\lambda \operatorname{sgn}(A) I & j I \\ j I & -\lambda \sqrt{|A|} B \sqrt{|A|} \end{bmatrix}\right). \end{aligned}$$

The matrix $\sqrt{|A|} B \sqrt{|A|}$ is symmetric, so it has n real eigenvalues $\omega_1, \dots, \omega_n$. On the other hand, these eigenvalues are exactly those of $|A| B$ because

$$\det[\sqrt{|A|} B \sqrt{|A|} - \omega I] = \det |A| \cdot \det[B - \omega |A|^{-1}] = \det[|A| B - \omega I].$$

Applying Lemma 3.7 with $\operatorname{sgn}(A) I$ and $\sqrt{|A|} B \sqrt{|A|}$ instead of A and B , respectively, we get

$$\begin{aligned} m^-(G_j((\lambda_0 + \varepsilon)L)) - m^-(G_j((\lambda_0 - \varepsilon)L)) &= \\ &= \operatorname{sgn}(A) \cdot \# \left\{ k \in \{1, \dots, n\} \mid \operatorname{sgn}(A) \omega_k > 0 \wedge \lambda_0 = \frac{j}{\sqrt{\operatorname{sgn}(A) \omega_k}} \right\}. \end{aligned}$$

But $\operatorname{sgn}(A) \omega_1, \dots, \operatorname{sgn}(A) \omega_n$ are the eigenvalues of $\operatorname{sgn}(A) |A| B = AB$, which completes the proof for A strictly positive or negative. If B is strictly positive or negative definite, consider

$$M = \begin{bmatrix} \sqrt{|B|} & 0 \\ 0 & (\sqrt{|B|})^{-1} \end{bmatrix}.$$

□

Note that the number s in the above lemma is well defined because we assume that $\sigma_+(AB) \neq \emptyset$. If A and B were nonsingular and of different signs then AB would be strictly negative definite, a contradiction. For example, if A is strictly positive and B is strictly negative then $\sigma(AB) = \sigma(\sqrt{A} B \sqrt{A})$ and for every $v \in \mathbb{R}^n$ we have

$$\langle \sqrt{A} B \sqrt{A} v, v \rangle = \langle B \sqrt{A} v, \sqrt{A} v \rangle < 0,$$

where $\langle \cdot, \cdot \rangle$ is an inner product in \mathbb{R}^n .

4. LOCAL BIFURCATIONS

In this section we formulate local bifurcation theorems for autonomous Hamiltonian systems (with block-diagonal Hessian of the Hamiltonian at a stationary point) in terms of the topological degree of the gradient of the Hamiltonian and eigenvalues of its Hessian computed at a stationary point.

Assume that $H \in C^2(\mathbb{R}^n \times \mathbb{R}^n, \mathbb{R})$ and that for fixed $x_0 \in \nabla H^{-1}(0)$, isolated in $\nabla H^{-1}(0)$, the Hessian of H at x_0 has the form

$$\nabla^2 H(x_0) = \begin{bmatrix} A & 0 \\ 0 & B \end{bmatrix} \quad (4.1)$$

for some $A, B \in \mathbb{S}(n, \mathbb{R})$.

Let $\sigma(A) = \{\alpha_1, \dots, \alpha_n\}$, $\sigma(B) = \{\beta_1, \dots, \beta_n\}$ and assume the convention of Remark 3.2 for the order of α_k, β_k .

In view of Theorem 2.2 we may suspect that the point (x_0, λ_0) is a bifurcation point of nontrivial solutions of (1.2) provided that $\lambda_0 \in \Lambda(\nabla^2 H(x_0))$. By Lemma 3.3 this means that

$$\lambda_0 = \frac{j_0}{\sqrt{\nu_0}}, \quad j_0 \in \mathbb{N}, \quad \nu_0 \in \sigma_+(AB). \quad (4.2)$$

If $AB = BA$ then every such a λ_0 can be written as $\lambda_0 = \frac{j_0}{\sqrt{\alpha_{k_0}\beta_{k_0}}}$ for some $k_0 \in \{1, \dots, n\}$ such that $\alpha_{k_0}\beta_{k_0} > 0$ (see Lemma 3.4). In the case of $AB \neq BA$ we cannot write λ_0 in this form in general, but it is possible to do it if $\alpha_{k_0}\beta_{k_0} > 0$ and $V_A(\alpha_{k_0}) \cap V_B(\beta_{k_0}) \neq \{0\}$ (see Lemma 3.5).

The following conditions will be used in theorems of this section.

(A1) $H \in C^2(\mathbb{R}^n \times \mathbb{R}^n, \mathbb{R})$, $x_0 \in \nabla H^{-1}(0)$ is isolated in $\nabla H^{-1}(0)$ and $i(\nabla H, x_0) \neq 0$,

(A2) $\nabla^2 H(x_0) = \begin{bmatrix} A & 0 \\ 0 & B \end{bmatrix}$, $A, B \in \mathbb{S}(n, \mathbb{R})$, $\nu_0 \in \sigma_+(AB)$,

(A3) $\lambda_0 = \frac{j_0}{\sqrt{\nu_0}}$, $j_0 \in \mathbb{N}$,

(A4) $AB = BA$ and $\sharp Y_j^+(\lambda_0) \neq \sharp Y_j^-(\lambda_0)$ for some $j \in \mathbb{N}$, where $Y_j^+(\lambda_0)$ and $Y_j^-(\lambda_0)$ are given by (3.2).

Theorem 4.1. *Assume that all conditions (A1)-(A4) are satisfied. Then (x_0, λ_0) is a branching point of nontrivial solutions of*

$$\begin{cases} \dot{x}(t) = \lambda J \nabla H(x(t)) \\ x(0) = x(2\pi), \end{cases}$$

where $\lambda \in (0, +\infty)$. Moreover, if $U = \Omega \times (a, b) \subset H_{2\pi}^1 \times (0, +\infty)$ is an open bounded neighbourhood of (x_0, λ_0) such that $\nabla H^{-1}(0) \cap \text{cl}(\Omega) = \{x_0\}$ and (x_0, λ_0) is the only trivial solution in $\text{cl}(U)$ satisfying (4.2) then $C(x_0, \lambda_0) \cap \partial U \neq \emptyset$.

Proof. Observe that $\lambda_0 \in \Lambda(\nabla^2 H(x_0))$ (see (A2), (A3), and (4.2)). Moreover, the assumptions of Lemma 3.7 are satisfied for $L = \nabla^2 H(x_0)$, according to (A2) and (A4). Using this lemma and equality (2.1) we obtain $\eta_j(x_0, \lambda_0) = i(\nabla H, x_0) \cdot (\sharp Y_j^+(\lambda_0) - \sharp Y_j^-(\lambda_0))$ and thus $\eta_j(x_0, \lambda_0) \neq 0$, by (A1) and (A4). Applying Theorem 2.3 we complete the proof. \square

The corresponding theorem concerning emanations of periodic trajectories from a stationary point can be formulated as follows.

Theorem 4.2. *Assume that all conditions (A1)-(A4) are satisfied. Then there exists a connected (with respect to Hausdorff metric) set of nonstationary periodic trajectories of*

$$\dot{x}(t) = J\nabla H(x(t))$$

emanating from x_0 with (not necessarily minimal) periods arbitrarily close to $\frac{j_0}{j} \frac{2\pi}{\sqrt{\nu_0}}$ for trajectories sufficiently close to x_0 . Moreover, if $j_0 = 1$ and $\sqrt{\frac{\nu}{\nu_0}} \notin \mathbb{N}$ for all $\nu \in \sigma_+(AB) \setminus \{\nu_0\}$ then there exists a connected set of nonstationary periodic trajectories emanating from x_0 with minimal periods arbitrarily close to $\frac{2\pi}{\sqrt{\nu_0}}$ for trajectories sufficiently close to x_0 .

Proof. Our claim is a consequence of Theorem 4.1, Lemma 3.7, and Remarks 2.5. To obtain minimal periods of trajectories observe that if $j_0 = 1$ and $\sqrt{\frac{\nu}{\nu_0}} \notin \mathbb{N}$ for all $\nu \in \sigma_+(AB) \setminus \{\nu_0\}$ then $\frac{\lambda_0}{\lambda} \notin \mathbb{N}$ for all $\lambda \in \Lambda(\nabla^2 H(x_0)) \setminus \{\lambda_0\}$. \square

Notice that if $j_0 = 1$ and $\sqrt{\frac{\nu}{\nu_0}} \notin \mathbb{N}$ for all $\nu \in \sigma_+(AB) \setminus \{\nu_0\}$ then condition (A4) in the above theorem can be satisfied only for $j = 1$, since in this case $\lambda_0 = \frac{1}{\sqrt{\nu_0}} = \frac{j}{\sqrt{\alpha_k \beta_k}}$ only for $j = j_0 = 1$ and $\nu_0 = \alpha_k \beta_k$, see (3.2).

The following theorem can be proved in the same way as Theorem 4.1, by using Lemma 3.8 instead of 3.7.

Theorem 4.3. *Suppose that conditions (A1), (A2), (A3) are satisfied and that for some $k_0 \in \{1, \dots, n\}$ we have*

$$\nu_0 = \alpha_{k_0} \beta_{k_0}, \quad \dim V_A(\alpha_{k_0}) \cap V_B(\beta_{k_0}) > \frac{1}{2} \mu(\nu_0). \quad (4.3)$$

Then the conclusion of Theorem 4.1 holds true.

Combining Theorem 4.3 and Lemma 3.8 (for $j_0 = 1$) with Remarks 2.5 we obtain the corresponding emanation result.

Theorem 4.4. *Suppose that assumptions (A1), (A2) are satisfied and that condition (4.3) is fulfilled for some $k_0 \in \{1, \dots, n\}$. Then there exists a connected (with respect to Hausdorff metric) set of nonstationary periodic trajectories of*

$$\dot{x}(t) = J\nabla H(x(t))$$

emanating from x_0 with (not necessarily minimal) periods arbitrarily close to $\frac{2\pi}{\sqrt{\nu_0}}$ for trajectories sufficiently close to x_0 . Moreover, if $\sqrt{\frac{\nu}{\nu_0}} \notin \mathbb{N}$ for all $\nu \in \sigma_+(AB) \setminus \{\nu_0\}$ then there exists a connected set of nonstationary periodic trajectories emanating from x_0 with minimal periods arbitrarily close to $\frac{2\pi}{\sqrt{\nu_0}}$ for trajectories sufficiently close to x_0 .

Similarly as above, application of Lemma 3.9 gives us the following.

Theorem 4.5. *Assume that conditions (A1), (A2) are satisfied and the multiplicity of ν_0 is odd. Then*

- (1) the conclusion of Theorem 4.1 is true for every λ_0 satisfying (A3),
- (2) the conclusion of Theorem 4.4 holds.

Finally, applying Lemma 3.10 (for $j = j_0$) we obtain

Theorem 4.6. *Let conditions (A1) and (A2) be fulfilled. Suppose that A or B is strictly positive or strictly negative definite. Then*

- (1) the conclusion of Theorem 4.1 is true for every λ_0 satisfying (A3),
- (2) the conclusion of Theorem 4.4 holds.

Corollary 4.7. *Let $H \in C^2(\mathbb{R}^n \times \mathbb{R}^n, \mathbb{R})$ admit a strict local minimum or maximum at x_0 and*

$$\nabla^2 H(x_0) = \begin{bmatrix} A & 0 \\ 0 & B \end{bmatrix}, \quad A, B \in \mathbb{S}(n, \mathbb{R}).$$

If A is nonsingular and $B \neq 0$, or B is nonsingular and $A \neq 0$, then

- (1) $\sigma_+(AB) \neq \emptyset$,
- (2) the conclusion of Theorem 4.1 holds for any $\lambda_0 = \frac{j_0}{\sqrt{\nu_0}}$, $j_0 \in \mathbb{N}$, $\nu_0 \in \sigma_+(AB)$,
- (3) the conclusion of Theorem 4.4 is true for every $\nu_0 \in \sigma_+(AB)$.

Proof. Since H admits a strict local minimum (maximum) at x_0 , we have $x_0 \in \nabla H^{-1}(0)$, x_0 is isolated in $\nabla H^{-1}(0)$, $i(\nabla H, x_0) = 1 \neq 0$ (see [1]) and $\nabla^2 H(x_0)$ is nonnegative (resp. nonpositive) definite, hence $\text{sgn}(A) = \text{sgn}(B)$. If, for example, A is nonsingular then

$$\sigma_+(AB) = \sigma_+(|A||B|) = \sigma_+(\sqrt{|A|}|B|\sqrt{|A|}) = \sigma_+(\sqrt{|A|}^t|B|\sqrt{|A|}).$$

But $B \neq 0$, therefore $\sigma_+(AB) \neq \emptyset$, in view of Sylvester's law of inertia. Applying Theorem 4.6 we complete the proof. \square

Example 4.8. Consider $H : \mathbb{R}^3 \times \mathbb{R}^3 \rightarrow \mathbb{R}$ given by the formula

$$H(x) = H(x_1, \dots, x_6) = x_1^2 + 2(x_2 - 1)^2 - x_3^2 + (x_4 + (x_3 - 2)^2)^6 + x_5^2 + (x_6 - 1)^2 - x_5 x_6.$$

In this case we have $\nabla H^{-1}(0) = \{(0, 1, 0, -4, \frac{2}{3}, \frac{4}{3})\}$ and $i(\nabla H, (0, 1, 0, -4, \frac{2}{3}, \frac{4}{3})) = -1$. (The last equality can be obtained by using an algorithm described in [8].) The Hessian $\nabla^2 H(0, 1, 0, -4, \frac{2}{3}, \frac{4}{3})$ has the block-diagonal form (4.1) with

$$A = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & -2 \end{bmatrix}, \quad B = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 2 & -1 \\ 0 & -1 & 2 \end{bmatrix}.$$

Moreover, $AB \neq BA$ and $\sigma(AB) = \{0, 2 + 2\sqrt{7}, 2 - 2\sqrt{7}\}$. According to Theorem 4.5, for every $j_0 \in \mathbb{N}$ the point $\left((0, 1, 0, -4, \frac{2}{3}, \frac{4}{3}), \frac{j_0}{\sqrt{2+2\sqrt{7}}}\right)$ is a branching point of nontrivial solutions of (1.2) and there exists a connected set of nonstationary periodic trajectories of (1.1) emanating from $(0, 1, 0, -4, \frac{2}{3}, \frac{4}{3})$ with minimal periods tending to $\frac{2\pi}{\sqrt{2+2\sqrt{7}}}$ at $(0, 1, 0, -4, \frac{2}{3}, \frac{4}{3})$.

Example 4.9. Define $H : \mathbb{R}^2 \times \mathbb{R}^2 \rightarrow \mathbb{R}$ as

$$H(x) = H(x_1, \dots, x_4) = x_1^2 + 2x_2^2 + x_3^2 + (x_4 + (x_3 - 1)^3)^4.$$

Observe that $\nabla H^{-1}(0) = \{(0, 0, 0, 1)\}$ and H admits a strict minimum at $(0, 0, 0, 1)$. Moreover, $\nabla^2 H(0, 0, 0, 1)$ is of the form (4.1) with

$$A = \begin{bmatrix} 2 & 0 \\ 0 & 4 \end{bmatrix}, \quad B = \begin{bmatrix} 2 & 0 \\ 0 & 0 \end{bmatrix}, \quad AB = BA = \begin{bmatrix} 4 & 0 \\ 0 & 0 \end{bmatrix}.$$

By Corollary 4.7, for every $j_0 \in \mathbb{N}$ the point $((0, 0, 0, 1), \frac{j_0}{2})$ is a branching point of nontrivial solutions of (1.2) and there exists a connected set of nonstationary periodic trajectories of (1.1) emanating from $(0, 0, 0, 1)$ with minimal periods tending to π at $(0, 0, 0, 1)$.

5. GLOBAL BIFURCATIONS

As in the previous section suppose that $H \in C^2(\mathbb{R}^n \times \mathbb{R}^n, \mathbb{R})$, but $\nabla H^{-1}(0)$ is finite. Assume also that for every fixed $x_0 \in \nabla H^{-1}(0)$ we have

$$\nabla^2 H(x_0) = L(x_0) = \begin{bmatrix} A(x_0) & 0 \\ 0 & B(x_0) \end{bmatrix}, \quad A(x_0), B(x_0) \in \mathbb{S}(n, \mathbb{R}). \quad (5.1)$$

According to Lemma 3.3, the set of parameters λ_0 which satisfy the bifurcation necessary condition from Theorem 2.2 is equal to $\Lambda(\nabla^2 H(x_0)) = \bigcup_{j \in \mathbb{N}} \Lambda_j(\nabla^2 H(x_0))$, where

$$\Lambda_j(\nabla^2 H(x_0)) = \left\{ \frac{j}{\sqrt{\nu}} \mid \nu \in \sigma_+(A(x_0)B(x_0)) \right\}.$$

Let

$$\sigma(A(x_0)) = \{\alpha_1(x_0), \dots, \alpha_n(x_0)\}, \quad \sigma(B(x_0)) = \{\beta_1(x_0), \dots, \beta_n(x_0)\}.$$

Repeat Remark 3.2 and equalities (3.2) for $A \equiv A(x_0)$, $B \equiv B(x_0)$, $\alpha_k \equiv \alpha_k(x_0)$, $\beta_k \equiv \beta_k(x_0)$ and $Y_j^\pm(\lambda_0) \equiv Y_j^\pm(x_0, \lambda_0)$.

Theorem 5.1. *Assume that $H \in C^2(\mathbb{R}^n \times \mathbb{R}^n, \mathbb{R})$, $\nabla H^{-1}(0)$ is finite, and that for every $\xi \in \nabla H^{-1}(0)$ we have*

$$\nabla^2 H(\xi) = \begin{bmatrix} A(\xi) & 0 \\ 0 & B(\xi) \end{bmatrix}, \quad A(\xi), B(\xi) \in \mathbb{S}(n, \mathbb{R}), \quad A(\xi)B(\xi) = B(\xi)A(\xi).$$

Let $x_0 \in \nabla H^{-1}(0)$ and $i(\nabla H, x_0) \neq 0$. If $\lambda_0 = \frac{j_0}{\sqrt{\nu_0}}$, $j_0 \in \mathbb{N}$, $\nu_0 \in \sigma_+(A(x_0)B(x_0))$, and $\sharp Y_l^+(x_0, \lambda_0) \neq \sharp Y_l^-(x_0, \lambda_0)$ for some $l \in \mathbb{N}$ then (x_0, λ_0) is a branching point of nontrivial solutions of

$$\begin{cases} \dot{x}(t) = \lambda J \nabla H(x(t)) \\ x(0) = x(2\pi), \end{cases}$$

where $\lambda \in (0, +\infty)$, and either

- (1) $C(x_0, \lambda_0)$ is unbounded in $H_{2\pi}^1 \times (0, +\infty)$ or

- (2) $C(x_0, \lambda_0)$ is bounded and, in addition, $C(x_0, \lambda_0) \cap \mathcal{T}(H) = \{(\xi_1, \eta_1), \dots, (\xi_m, \eta_m)\}$ for some $m \in \mathbb{N}$, $\xi_i \in \nabla H^{-1}(0)$, $\eta_i = \frac{j_i}{\sqrt{\omega_i}}$, $j_i \in \mathbb{N}$, $\omega_i \in \sigma_+(A(\xi_i)B(\xi_i))$, $i = 1, \dots, m$, and for any $j \in \mathbb{N}$ we have

$$\sum_{i=1}^m i(\nabla H, \xi_i) \cdot (\#Y_j^+(\xi_i, \eta_i) - \#Y_j^-(\xi_i, \eta_i)) = 0.$$

Proof. According to Theorem 2.4, the sum of bifurcation indices of the points from $C(x_0, \lambda_0) \cap \mathcal{T}(H)$ is equal to Θ . But $\eta_j(\xi_i, \eta_i) = i(\nabla H, \xi_i) \cdot (\#Y_j^+(\xi_i, \eta_i) - \#Y_j^-(\xi_i, \eta_i))$ for $i = 1, \dots, m$, in view of equality (2.1) and Lemma 3.7. \square

Theorem 5.2. Assume that $H \in C^2(\mathbb{R}^n \times \mathbb{R}^n, \mathbb{R})$, $\nabla H^{-1}(0)$ is finite, and that for every $\xi \in \nabla H^{-1}(0)$ we have

$$\nabla^2 H(\xi) = \begin{bmatrix} A(\xi) & 0 \\ 0 & B(\xi) \end{bmatrix}, \quad A(\xi), B(\xi) \in \mathbb{S}(n, \mathbb{R}),$$

where $A(\xi)$ or $B(\xi)$ is strictly positive or strictly negative definite. Let $x_0 \in \nabla H^{-1}(0)$ and $i(\nabla H, x_0) \neq 0$. If $\lambda_0 = \frac{j_0}{\sqrt{\nu_0}}$, $j_0 \in \mathbb{N}$, $\nu_0 \in \sigma_+(A(x_0)B(x_0))$ then (x_0, λ_0) is a branching point of nontrivial solutions of

$$\begin{cases} \dot{x}(t) = \lambda J \nabla H(x(t)) \\ x(0) = x(2\pi), \end{cases}$$

where $\lambda \in (0, +\infty)$, and either

- (1) $C(x_0, \lambda_0)$ is unbounded in $H_{2\pi}^1 \times (0, +\infty)$ or
- (2) $C(x_0, \lambda_0)$ is bounded and, in addition, $C(x_0, \lambda_0) \cap \mathcal{T}(H)$ is finite, for any $j \in \mathbb{N}$ we have

$$C(x_0, \lambda_0) \cap \bigcup_{\xi \in \nabla H^{-1}(0)} \{\xi\} \times \Lambda_j(\nabla^2 H(\xi)) = \left\{ \left(\xi_1, \frac{j}{\sqrt{\omega_1}} \right), \dots, \left(\xi_m, \frac{j}{\sqrt{\omega_m}} \right) \right\}$$

for some $m \in \mathbb{N}$, $\xi_i \in \nabla H^{-1}(0)$, $\omega_i \in \sigma_+(A(\xi_i)B(\xi_i))$, $i = 1, \dots, m$ (if the above intersection is nonempty), and

$$\sum_{i=1}^m i(\nabla H, \xi_i) \cdot s(\xi_i) \cdot \mu(\omega_i) = 0,$$

where $s(\xi_i) = 1$ if $A(\xi_i)$ or $B(\xi_i)$ is strictly positive definite and $s(\xi_i) = -1$ if $A(\xi_i)$ or $B(\xi_i)$ is strictly negative definite.

Proof. According to Theorem 2.4, the sum of bifurcation indices of the points from $C(x_0, \lambda_0) \cap \mathcal{T}(H)$ is equal to Θ , i.e. it vanishes at every coordinate. In view of the equality (2.1) and Lemma 3.10 the set

$$C(x_0, \lambda_0) \cap \bigcup_{\xi \in \nabla H^{-1}(0)} \{\xi\} \times \Lambda_j(\nabla^2 H(\xi))$$

consists of those points from $C(x_0, \lambda_0) \cap \mathcal{T}(H)$ for which the j th coordinate of the bifurcation index η can be nonzero. Namely, $\eta_j(\xi_i, \frac{j}{\sqrt{\omega_i}}) = i(\nabla H, \xi_i) \cdot s(\xi_i) \cdot \mu(\omega_i)$ for $i = 1, \dots, m$. \square

Corollary 5.3. *If the assumptions of Theorem 5.2 are satisfied and $\nabla H^{-1}(0) = \{x_0\}$ then $C(x_0, \lambda_0)$ is unbounded in $H_{2\pi}^1 \times (0, +\infty)$.*

Example 5.4. Let H be such as in Example 4.9. Then A is strictly positive definite and $i(\nabla H, (0, 0, 0, 1)) = 1$. In view of Corollary 5.3, for every $j_0 \in \mathbb{N}$ the connected branch $C((0, 0, 0, 1), \frac{j_0}{2})$ bifurcating from $((0, 0, 0, 1), \frac{j_0}{2})$ is unbounded in $H_{2\pi}^1 \times (0, +\infty)$.

From now on we assume that $H \in C^2(\mathbb{R}^n \times \mathbb{R}^n, \mathbb{R})$ satisfies the assumptions of Theorem 5.2. Let $s(\xi)$ be such as in that theorem and set

$$\begin{aligned} S_+(H) &= \{\xi \in \nabla H^{-1}(0) \mid i(\nabla H, \xi) \cdot s(\xi) > 0\}, \\ S_-(H) &= \{\xi \in \nabla H^{-1}(0) \mid i(\nabla H, \xi) \cdot s(\xi) < 0\}, \\ p(H) &= \{(\xi, \omega) \mid \xi \in S_+(H), \omega \in \sigma_+(A(\xi)B(\xi))\}, \\ n(H) &= \{(\xi, \omega) \mid \xi \in S_-(H), \omega \in \sigma_+(A(\xi)B(\xi))\}, \\ \mathcal{E}(H) &= \sum_{(\xi, \omega) \in p(H) \cup n(H)} i(\nabla H, \xi) \cdot s(\xi) \cdot \mu(\omega). \end{aligned}$$

Let us formulate further corollaries to Theorem 5.2. If $A(\xi) = I$ for all $\xi \in \nabla H^{-1}(0)$ then they imply corresponding corollaries from [12].

Corollary 5.5. *If $\mathcal{E}(H) \neq 0$ then for every $j \in \mathbb{N}$ there exists $(\xi, \omega) \in p(H) \cup n(H)$ such that $C(\xi, \frac{j}{\sqrt{\omega}})$ is unbounded in $H_{2\pi}^1 \times (0, +\infty)$. Moreover, if additionally $p(H) = \emptyset$ or $n(H) = \emptyset$ then $C(\xi, \frac{j}{\sqrt{\omega}})$ are unbounded for all $j \in \mathbb{N}$, $(\xi, \omega) \in p(H) \cup n(H)$.*

Proof. Fix $j \in \mathbb{N}$ and observe that if for all $(\xi, \omega) \in p(H) \cup n(H)$ sets $C(\xi, \frac{j}{\sqrt{\omega}})$ were bounded then the sum of $i(\nabla H, \xi) \cdot s(\xi) \cdot \mu(\omega)$ over $p(H) \cup n(H)$ would be equal to 0, a contradiction. \square

Corollary 5.6. *If $\mathcal{E}(H) \neq 0$ and $|i(\nabla H, \xi) \cdot \mu(\omega)| = c = \text{const}$ for all $(\xi, \omega) \in p(H) \cup n(H)$ then for every $j \in \mathbb{N}$ sets $C(\xi, \frac{j}{\sqrt{\omega}})$ are unbounded in $H_{2\pi}^1 \times (0, +\infty)$ for at least $|\sharp p(H) - \sharp n(H)|$ of $(\xi, \omega) \in p(H) \cup n(H)$.*

Proof. Assume, for example, that $\sharp p(H) > \sharp n(H)$. Denote by Z_p (Z_n) the set of such points $(\xi, \omega) \in p(H)$ (resp. $(\xi, \omega) \in n(H)$) that $C(\xi, \frac{j}{\sqrt{\omega}})$ is bounded. The sum of $i(\nabla H, \xi) \cdot s(\xi) \cdot \mu(\omega)$ over $Z_p \cup Z_n$ is equal to 0, in view of Theorem 5.2. Thus $\sharp Z_p = \sharp Z_n$. But $\sharp Z_n \leq \sharp n(H)$, hence the number of $(\xi, \omega) \in p(H) \cup n(H)$ for which $C(\xi, \frac{j}{\sqrt{\omega}})$ is unbounded is equal to $(\sharp p(H) - \sharp Z_p) + (\sharp n(H) - \sharp Z_n) = \sharp p(H) + \sharp n(H) - 2\sharp Z_n \geq \sharp p(H) + \sharp n(H) - 2\sharp n(H) = \sharp p(H) - \sharp n(H)$. \square

Obviously, unbounded sets $C(\xi, \frac{j}{\sqrt{\omega}})$ from the above corollary need not be different for different $(\xi, \omega) \in p(H) \cup n(H)$.

Corollary 5.7. *Suppose that $\deg(\nabla H, U, 0) \neq 0$ for some open and bounded $U \subset \mathbb{R}^{2n}$ such that $\nabla H^{-1}(0) \subset U$. Let $p(H) \cup n(H) \neq \emptyset$. If $\sharp \sigma_+(\nabla^2 H(\xi)) = b = \text{const}$, $s(\xi) = s = \text{const}$ and $\mu(\omega) = m = \text{const}$ for all $(\xi, \omega) \in p(H) \cup n(H)$ then for every $j \in \mathbb{N}$ there exists $(\xi, \omega) \in p(H) \cup n(H)$ such that $C(\xi, \frac{j}{\sqrt{\omega}})$ is unbounded in $H_{2\pi}^1 \times (0, +\infty)$.*

Proof. Suppose, contrary to our claim, that for some $j \in \mathbb{N}$ sets $C(\xi, \frac{j}{\sqrt{\omega}})$ are bounded for all $(\xi, \omega) \in p(H) \cup n(H)$. According to Theorem 5.2 the sum of $i(\nabla H, \xi) \cdot s(\xi) \cdot \mu(\omega)$ over these points is equal to 0. On the other hand this sum is equal to

$$b \cdot s \cdot m \cdot \sum_{\xi \in \nabla H^{-1}(0)} i(\nabla H, \xi) = b \cdot s \cdot m \cdot \deg(\nabla H, U, 0) \neq 0,$$

a contradiction. □

Note that the condition $\deg(\nabla H, U, 0) \neq 0$ in the above corollary is satisfied if $H(x) \rightarrow +\infty$ as $|x| \rightarrow +\infty$ (see [1]). For strictly convex H the last condition is equivalent to the condition $\nabla H^{-1}(0) \neq \emptyset$ (see [10]).

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